

IML 2003-12
math-ph/0310062

q -deformation of $z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$

Ctirad Klimčík

Institute de mathématiques de Luminy,
163, Avenue de Luminy, 13288 Marseille, France

Abstract

We construct the action of the quantum double of $U_q(su(2))$ on the standard Podleś sphere and interpret it as the quantum projective formula generalizing to the q -deformed setting the action of the Lorentz group of global conformal transformations on the ordinary Riemann sphere.

1 Introduction

As it is well-known, the Lorentz group $SL(2, \mathbf{C})$ naturally acts on the Riemann sphere S^2 by the conformal transformations

$$z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbf{C}, \quad \alpha\delta - \beta\gamma = 1. \quad (1)$$

The q -deformation of S^2 is referred to as the [Podleś] sphere. One of the goals of this note is to find the corresponding q -deformation of the conformal transformations (1). We note that the restriction $\bar{\alpha} = \delta, \bar{\gamma} = -\beta$ gives the action of the group $SU(2)$ on S^2 which just corresponds to the standard geometrical rotations of the two-sphere embedded into three-dimensional Euclidean space. The q -deformed version of this $SU(2)$ action was studied in detail by [Podleś]. However, to our best knowledge, the q -conformal action of the full q -Lorentz group on the Podleś sphere has not yet been reported.

Recall, that the algebra of functions $\text{Fun}(S^2)$ on ordinary two-sphere can be viewed as the algebra of those functions on the group $SU(2)$ which are invariant with respect to the right action of the maximal torus $U(1)$ on $SU(2)$. This gives the dual description of the coset $SU(2)/U(1) \equiv S^2$. The group $SU(2)$ acts naturally from the left on this coset and this action can be extended to the action (1) of the Lorentz group $SL(2, \mathbf{C})$ on S^2 since we have a well-known identification $SU(2)/U(1) \equiv SL(2, \mathbf{C})/B$ with B being the Borel subgroup (consisting of uppertriangular matrices) of $SL(2, \mathbf{C})$.

In order to construct the q -deformation of the picture just described, we take some inspiration from the theory of Poisson-Lie groups (see [Semenov-Tian-Shansky, Klimčík] for the elements). There is the Iwasawa decomposition $SL(2, \mathbf{C}) = SU(2)AN$ of the Lorentz group where AN is the subgroup of uppertriangular complex 2×2 matrices with real positive numbers on diagonal and unit determinant. AN turns out to be the dual Poisson Lie group of $SU(2)$ and $SL(2, \mathbf{C})$ is the Drinfeld double of $SU(2)$ in the Poisson-Lie sense of this word. Now the Drinfeld double $SL(2, \mathbf{C})$ acts on $\text{Fun}(SU(2))$: the action of its subgroup $SU(2)$ is induced just by the left multiplication of $SU(2)$ on itself and the subgroup AN acts by the so-called dressing transformations. This action of the $SL(2, \mathbf{C})$ on $\text{Fun}(SU(2))$ descends to $\text{Fun}(SU(2)/U(1) (\equiv S^2))$ and it turns out to be given by the projective action (1), where z is the standard complex coordinate on the Riemann sphere.

The theory of Poisson-Lie groups is a sort of the semiclassical limit of the theory of q -deformed Hopf algebras for the deformation parameter q approaching 1. Many Poisson-Lie concepts can be directly generalized to the Hopf algebra setting like e.g. the Drinfeld double or the dressing transformations. In particular, the Poisson-Lie concept of the duality translates into the famous [Drinfeld] duality in the world of Hopf algebras. Having in mind the parallels between the Poisson-Lie and the Hopf worlds, it is not difficult to find the q -deformation of the projective formula. We proceed as follows.

The Podleś sphere $\text{Fun}_q(S^2)$ is a one-parameter deformation of the algebra $\text{Fun}(S^2)$. It is generated by the $U(1)$ right-invariant elements of the quantum group $\text{Fun}_q(SU(2))$. The deformed envelopping algebra $U_q(\mathfrak{su}(2))$ naturally acts on the deformed $\text{Fun}_q(SU(2))$. (This corresponds to the left action of $SU(2)$ on $\text{Fun}(SU(2))$ just described above). The Hopf dual $U_q(\mathfrak{an})$ of $U_q(\mathfrak{su}(2))$ acts on $\text{Fun}_q(SU(2))$ in the Hopf-dressing way. (This corresponds to the dressing action of AN on $\text{Fun}(SU(2))$). As noted by [Korogodsky], the Hopf analogue of the dressing action of $U_q(\mathfrak{an})$ on $\text{Fun}_q(SU(2))$ is the adjoint action of the Hopf algebra $\text{Fun}_q(SU(2))$ on itself. This statement is consistent due to the Drinfeld duality isomorphism between $U_q(\mathfrak{an})$ and $\text{Fun}_q(SU(2))$. The respective actions of $U_q(\mathfrak{an})$ and $U_q(\mathfrak{su}(2))$ on $\text{Fun}_q(SU(2))$ combine to the action of the Drinfeld double $D(U_q(\mathfrak{su}(2)))$ on $\text{Fun}_q(SU(2))$. This Drinfeld double is nothing but the q -Lorentz group (see [Podleś & Woronowicz]) and the only consistency check of the construction consists in verifying that the action of the q -Lorentz group descends from $\text{Fun}_q(SU(2))$ on $\text{Fun}_q(S^2)$. It turns out to be the case and thus we obtain the q -deformation of the projective formula [1].

In section 2, we describe the action of the quantum double $D(U_q(\mathfrak{su}(2)))$ on the Podleś sphere and in section 3 we show that it leads to the projective formula (1) in the limit $q \rightarrow 1$. We finish with a short outlook.

2 Action of the Drinfeld double $D(U_q(\mathfrak{su}(2)))$ on the Podleś sphere

First we recall some relevant facts concerning (the $*$ -actions of) the Drinfeld double. The reader can mostly find them also in [Majid], however, our exposition between Eqs. (9) and (11) is original.

Thus let H be a Hopf algebra, \tilde{H} its dual and H^{cop} the coopposite Hopf algebra of H . The Drinfeld double $D(H)$ is another Hopf algebra which is generated by its two sub-Hopf algebras H^{cop} and \tilde{H} . The coalgebra structure of $D(H)$ is just that of $H^{cop} \otimes \tilde{H}$, the antipode S_D is given by

$$\begin{aligned} S_D(U \otimes f) &\equiv (1 \otimes Sf)(S^{cop}U \otimes 1) = \\ &= (S^{cop}U)'' \otimes (Sf)'' < (S^{cop}U)', (Sf)' > < S^{cop}(S^{cop}U)''', (Sf)''' > \end{aligned} \quad (2)$$

and the product is defined by the following cross relations [Majid]:

$$\langle U', f' \rangle (U'' \otimes 1)(1 \otimes f'') = \langle U'', f'' \rangle (1 \otimes f')(U' \otimes 1). \quad (3)$$

Here $U \in H^{cop}$, $f \in \tilde{H}$, $\langle \cdot, \cdot \rangle$ is the duality pairing between H and \tilde{H} and we use the Sweedler notation for the coproduct

$$\Delta^{cop}(U) = \sum_p U'_p \otimes U''_p \equiv U' \otimes U'', \quad \Delta(f) = f' \otimes f''. \quad (4)$$

The formula (3) is particularly useful if we know the generators and their relations for both algebras H^{cop} and \tilde{H} separately. The set of relations for the algebra structure of $D(H)$ can be then directly obtained from (3) and (4).

If, moreover, H and \tilde{H} are equipped with a compatible star structures, then the quantum double $D(H)$ can be also naturally made a $*$ -Hopf algebra. Recall that a star $*$ on H is a antilinear antihomomorphism of H satisfying $S*S* = Id$, $*^2 = Id$, $(*\otimes*)\Delta = \Delta*$ and $*\varepsilon = \varepsilon*$. The standard compatibility relation (cf.[Majid]) between the stars on H and \tilde{H} reads

$$< U^*, f > = \overline{< U, (Sf)^* >}, \quad U \in H, f \in \tilde{H}. \quad (5)$$

The explicit formula for the star $*$ on $D(H)$ is then uniquely determined as follows

$$\begin{aligned} (U \otimes f)^* &\equiv (1 \otimes f^*)(U^* \otimes 1) = \\ &= U^{*''} \otimes f^{*''} < U^{*'}, f^{*'} > < S^{cop}U^{*''''}, f^{*''''} >, \quad U \in H^{cop}, f \in \tilde{H}, \end{aligned} \quad (6)$$

where the star on H^{cop} is the same as that on H .

The algebras H^{cop} and \tilde{H} act (from the left) on \tilde{H} respectively as

$$U \triangleright h = \langle S^{cop}(U), h' \rangle h'', \quad U \in H^{cop}, h \in \tilde{H}, \quad (7a)$$

$$f \triangleright h = f' h S(f''), \quad f, h \in \tilde{H}. \quad (7b)$$

We note that $S^{cop} = S^{-1}$, where S is the antipode of H . Using the basic axioms of Hopf algebras, it is easy to check that the definitions (7ab) imply

$$\langle U', f' \rangle U'' \triangleright (f'' \triangleright h) = \langle U'', f'' \rangle f' \triangleright (U' \triangleright h). \quad U \in H^{cop}, \quad f, h \in \tilde{H}.$$

By comparing with the defining relation (3), this means that (7ab) describes in fact the left action of the quantum double $D(H)$ on \tilde{H} . Explicitly:

$$(U \otimes f) \triangleright h \equiv U \triangleright (f \triangleright h), \quad U \in H^{cop}, \quad f, h \in \tilde{H} \quad (8)$$

It can be also directly checked (with the help of the condition (5)), that this action is compatible with the algebra structure of \tilde{H} and with the $*$ -structure on \tilde{H} . Explicitly:

$$x \triangleright (fh) = (x' \triangleright f)(x'' \triangleright h), \quad x \in D(H), f, h \in \tilde{H}, \quad (9a)$$

$$(x \triangleright f)^* = (S_D(x))^* \triangleright f^*, \quad x \in D(H), f \in \tilde{H}. \quad (9b)$$

Now let $k \in H^{cop}$ be a group-like selfadjoint element, i.e. $k^* = k$, $\Delta^{cop}k = k \otimes k$, $\varepsilon(k) = 1$. We can then define a linear space A consisting of invariant elements of \tilde{H} with respect to the right action of k and $S(k)$ on \tilde{H} :

$$A = \{f \in \tilde{H}, \quad \langle f'', k \rangle f' \equiv f \triangleleft k = f, \quad f \triangleleft S(k) = f\}. \quad (10)$$

We have for $f, g \in A$

$$(fg) \triangleleft k = \langle f'' g'', k \rangle f' g' = \langle f'', k \rangle \langle g'', k \rangle f' g' = fg$$

and, in the same way, $(fg) \triangleleft S(k) = 1$ which means that A is the subalgebra of \tilde{H} . We obtain easily also the $*$ -stability of A , since for $f \in A$ we have

$$f^* \triangleleft k = \langle f^{*''}, k \rangle f^{*'} = (\langle f'', (S(k))^* \rangle f')^* = (f \triangleleft S(k))^* = f^*$$

and, in the same way, $f^* \triangleleft S(k) = (f \triangleleft k)^* = f^*$.

It is not difficult to prove that A is also stable with respect to the action (7ab) of the quantum double $D(H)$ on \tilde{H} . Indeed, we have for the H^{cop} action (7a):

$$U \triangleright (f \triangleleft k) = \langle S^{cop}(U), f' \rangle f'' \triangleleft f''' \triangleleft k = (U \triangleright f) \triangleleft k, \quad U \in H^{cop}, f \in \tilde{H}.$$

The proof of stability for \tilde{H} action (7b) is slightly more involved:

$$\begin{aligned} (h \triangleright f) \triangleleft k &= \langle (h' f S(h''))'', k \rangle ((h' f S(h''))' = \langle h'' f'' S h''', k \rangle h' f' S h'''' = \\ &= \langle h'', k \rangle \langle f'', k \rangle \langle S h''', k \rangle h' f' S h'''' = \langle h'', k S(k) \rangle \langle f'', k \rangle h' f' S h'''' = \\ &= \langle f'', k \rangle h' f' S h'' = h \triangleright (f \triangleleft k), \quad h, f \in \tilde{H}. \end{aligned}$$

The same formulae hold true upon replacing $k \rightarrow S(k)$.

In the context of our paper, the $*$ -Hopf algebra H will be the standard deformation $U_q(su(2))$ of $U(su(2))$, \tilde{H} will be the corresponding dual deformation $\text{Fun}_q(SU(2))$ of $\text{Fun}(SU(2))$ and A will be the Podleś sphere $\text{Fun}_q(S^2)$. For the sake of mathematical rigour, we should pay attention to the fact that the notion of the dual Hopf algebra needs some clarification in the infinite-dimensional case. Actually, $U_q(su(2))$ and $\text{Fun}_q(SU(2))$ are in duality in the sense of chapters V.7 and VII.4 of the book of [Kassel]. The general Drinfeld double formulae (2) - (10) then work with this notion of duality with the bilinear pairing given by Eq. (12d).

For the description of the Hopf algebras $\text{Fun}_q(SU(2))$ and $U_q(su(2))$, we use the conventions of [Dąbrowski & Sitarz] and [Dąbrowski et al]. Thus let $q \neq 1$ be a real positive number and denote $\text{Fun}_q(SU(2))$ a $*$ -Hopf algebra generated by a and b , subject to relations

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1, \quad (11a)$$

equipped with a coproduct

$$\Delta a = a \otimes a - qb \otimes b^*, \quad \Delta b = b \otimes a^* + a \otimes b, \quad (11b)$$

a counit $\varepsilon(a) = 1$, $\varepsilon(b) = 0$ and an antipode

$$Sa = a^*, \quad Sa^* = a, \quad Sb = -qb, \quad Sb^* = -q^{-1}b^*. \quad (11c)$$

The algebra $\text{Fun}_q(SU(2))$ is thus well defined but it is perhaps useful to comment its name. As everywhere in this paper, the symbol $\text{Fun}_q(M)$ indicates the deformation of the algebra of certain class of functions on the ordinary manifold M . If the manifold M is the Lie group then the typical functions in this class are the matrix elements of the finite-dimensional representations of this group (cf. [Levendorskii & Soibelman]).

The $*$ -Hopf algebra $U_q(su(2))$ is generated by elements e and (invertible self-adjoint) k , subject to relations

$$ek = qke, \quad k^2 - k^{-2} = (q - q^{-1})(e^*e - ee^*), \quad (12a)$$

equipped with a coproduct

$$\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad (12b)$$

a counit $\varepsilon(k) = 1$, $\varepsilon(e) = 0$ and an antipode

$$Se = -q^{-1}e, \quad Se^* = -qe^*, \quad Sk = k^{-1}. \quad (12c)$$

The (non-degenerate) duality pairing between $U_q(su(2))$ and $\text{Fun}_q(SU(2))$ is given by the two-dimensional representation of $U_q(su(2))$, i.e.

$$\langle k, a \rangle = q^{\frac{1}{2}}, \quad \langle k, a^* \rangle = q^{-\frac{1}{2}}, \quad \langle e, -qb^* \rangle = \langle e^*, b \rangle = 1 \quad (12d)$$

with all other couples of generators pairing to 0. It is easy to verify that the star structures on $\text{Fun}_q(SU(2))$ and $U_q(su(2))$ are compatible in the sense of Eq. (5).

The Podleś sphere is the algebra $\text{Fun}_q(S^2)$ viewed as the subalgebra of $\text{Fun}_q(SU(2))$ of right invariant elements with respect to the action of the self-adjoint group-like elements k and k^{-1} (cf. (10)). It is generated by

$$B = ab, \quad B^* = b^*a^*, \quad A = bb^*,$$

obeying the following relations

$$AB = q^2BA, \quad AB^* = q^{-2}B^*A, \quad BB^* = q^{-2}A(1-A), \quad B^*B = A(1-q^2A).$$

The action of the q -Lorentz group $D(U_q(su(2)))$ on $\text{Fun}_q(S^2) \subset \text{Fun}_q(SU(2))$ is described by the formulae (7ab). We obtain explicitly

$$\begin{aligned} k \triangleright B &= q^{-1}B, \quad k \triangleright B^* = qB^*, \quad k \triangleright A = A, \\ k^{-1} \triangleright B &= qB, \quad k^{-1} \triangleright B^* = q^{-1}B^*, \quad k^{-1} \triangleright A = A, \\ e \triangleright B &= 0, \quad e \triangleright B^* = q^{-\frac{1}{2}} - (q^{\frac{3}{2}} + q^{-\frac{1}{2}})A, \quad e \triangleright A = q^{\frac{1}{2}}B, \end{aligned}$$

$$\begin{aligned}
e^* \triangleright B &= -q^{-\frac{3}{2}} + (q^{\frac{1}{2}} + q^{-\frac{3}{2}})A, & e^* \triangleright B^* &= 0, & e^* \triangleright A &= -q^{-\frac{1}{2}}B^*, & (13a) \\
a \triangleright B &= q^{-1}B + (q - q^{-1})BA, & a \triangleright B^* &= q^{-1}B^* + (q - q^{-1})AB^*, \\
a \triangleright A &= q^{-2}A + (1 - q^{-2})A^2, \\
a^* \triangleright B &= qB + (q - q^3)AB, & a^* \triangleright B^* &= qB^* + (q - q^3)B^*A, \\
a^* \triangleright A &= q^2A + (q^2 - q^4)A^2, \\
b \triangleright B &= (q^2 - 1)B^2, & b \triangleright B^* &= (1 - q^2)A^2, & b \triangleright A &= (q^3 - q)BA, \\
b^* \triangleright B &= (q - q^{-1})A^2, & b^* \triangleright B^* &= -(q - q^{-1})B^{*2}, & b^* \triangleright A &= (1 - q^2)AB^*. & (13b)
\end{aligned}$$

We note, that the notion of $*$ -structure is crucial for our paper because the group $SL(2, \mathbf{C})$ (in the context of the conformal transformations acting on the Riemann sphere) is viewed as the real group. It is this fact which is the starting point of our strategy to deform the projective formula (1), since the real group $SL(2, \mathbf{C})$ is the Poisson-Lie Drinfeld double of the group $SU(2)$. The concept of reality in the deformed Hopf picture is encoded in the $*$ -structure. Thus we need a star $*$ on our quantum double $D(U_q(su(2))) = SL_q(2, \mathbf{C})$. It is in fact given by the formula (6) uniquely in terms of the standard stars on $U_q(su(2))$ and $\text{Fun}_q(SU(2))$ (see [Majid, Dąbrowski et al.]). The star-compatible action of the $*$ -Hopf algebra $D(U_q(su(2)))$ on the $*$ -algebra $\text{Fun}_q(SU(2))$ (and on its subalgebra $\text{Fun}_q(S^2)$) is the q -deformed version of the statement that the real group $SL(2, \mathbf{C})$ acts on the real algebra $\text{Fun}(SU(2))$ and on its subalgebra $\text{Fun}(S^2)$.

3 The limit $q \rightarrow 1$

In this section, we want to show that the action (7ab) of the quantum double $D(U_q(su(2)))$ on $\text{Fun}_q(S^2)$ described explicitly by the formulae (13ab) gives in the limit $q \rightarrow 1$ the same result as the action of the group $SL(2, \mathbf{C})$ on $\text{Fun}(S^2)$ induced by the projective formula (1). First of all, the limit $q \rightarrow 1$ of $\text{Fun}_q(S^2)$ gives the commutative algebra of complex functions on the sphere S^2 , generated by

$$\mathcal{B} = \frac{z}{z\bar{z} + 1}, \quad \mathcal{B}^* = \frac{\bar{z}}{z\bar{z} + 1}, \quad \mathcal{A} = \frac{1}{z\bar{z} + 1},$$

where z is the standard complex coordinate on the Riemann sphere given by the stereographic projection.

The subgroup $SU(2)$ of $SL(2, \mathbf{C})$ acts on S^2 via formula (1)

$$z \rightarrow \frac{\alpha z + \beta}{-\beta z + \bar{\alpha}}, \quad \bar{z} \rightarrow \frac{\bar{\alpha} \bar{z} + \bar{\beta}}{-\beta \bar{z} + \alpha}.$$

Its Lie algebra $Lie(SU(2))$ therefore acts on $\text{Fun}(S^2)$ via three vector fields \mathcal{R}_j , $j = 1, 2, 3$:

$$\mathcal{R}_3 = i(z\partial_z - \bar{z}\partial_{\bar{z}}), \quad \mathcal{R}_1 + i\mathcal{R}_2 = i(\partial_z + \bar{z}^2\partial_{\bar{z}}), \quad -\mathcal{R}_1 + i\mathcal{R}_2 = i(z^2\partial_z + \partial_{\bar{z}}).$$

The subgroup AN of $SL(2, \mathbf{C})$ is formed by complex upper-triangular 2×2 -matrices with real positive numbers on the diagonal. Its action on S^2 is obtained from the projective formula (1) for the following choice of parameters: $\gamma = 0$, $\text{Im}\alpha = 0$, $\text{Re}\alpha > 0$ and β an arbitrary complex number. Thus

$$z \rightarrow \alpha(\alpha z + \beta), \quad \bar{z} \rightarrow \alpha(\alpha \bar{z} + \bar{\beta}).$$

The Lie algebra $Lie(AN)$ therefore acts on $\text{Fun}(S^2)$ via three vector fields \mathcal{T}_j , $j = 0, 1, 2$:

$$\mathcal{T}_0 = z\partial_z + \bar{z}\partial_{\bar{z}}, \quad \mathcal{T}_2 + i\mathcal{T}_1 = -2\partial_{\bar{z}}, \quad -\mathcal{T}_2 + i\mathcal{T}_1 = 2\partial_z.$$

It is now straightforward to calculate

$$\mathcal{R}_3\mathcal{B} = i\mathcal{B}, \quad \mathcal{R}_3\mathcal{B}^* = -i\mathcal{B}^*, \quad \mathcal{R}_3\mathcal{A} = 0.$$

$$\begin{aligned} (\mathcal{R}_1 + i\mathcal{R}_2)\mathcal{B} &= i(2\mathcal{A} - 1), & (\mathcal{R}_1 + i\mathcal{R}_2)\mathcal{B}^* &= 0, & (\mathcal{R}_1 + i\mathcal{R}_2)\mathcal{A} &= -i\mathcal{B}^*, \\ (-\mathcal{R}_1 + i\mathcal{R}_2)\mathcal{B} &= 0, & (-\mathcal{R}_1 + i\mathcal{R}_2)\mathcal{B}^* &= i(2\mathcal{A} - 1), & (-\mathcal{R}_1 + i\mathcal{R}_2)\mathcal{A} &= -i\mathcal{B}, \end{aligned} \tag{14a}$$

$$\begin{aligned} \mathcal{T}_0\mathcal{B} &= \mathcal{B}(2\mathcal{A} - 1), & \mathcal{T}_0\mathcal{B}^* &= \mathcal{B}^*(2\mathcal{A} - 1), & \mathcal{T}_0\mathcal{A} &= 2\mathcal{A}(\mathcal{A} - 1), \\ (\mathcal{T}_2 + i\mathcal{T}_1)\mathcal{B} &= 2\mathcal{B}^2, & (\mathcal{T}_2 + i\mathcal{T}_1)\mathcal{B}^* &= -2\mathcal{A}^2, & (\mathcal{T}_2 + i\mathcal{T}_1)\mathcal{A} &= 2\mathcal{A}\mathcal{B}, \\ (-\mathcal{T}_2 + i\mathcal{T}_1)\mathcal{B} &= 2\mathcal{A}^2, & (-\mathcal{T}_2 + i\mathcal{T}_1)\mathcal{B}^* &= -2\mathcal{B}^{*2}, & (-\mathcal{T}_2 + i\mathcal{T}_1)\mathcal{A} &= -2\mathcal{A}\mathcal{B}^*. \end{aligned} \tag{14b}$$

We recall, that the formulae (14ab) describe the infinitesimal projective action (1) of the Lie Algebra $Lie(SL(2, \mathbf{C}))$ on $\text{Fun}(S^2)$. We wish to show that they can be obtained from the formulae (13ab) in the limit $q \rightarrow 1$.

In the limit $q \rightarrow 1$, the Hopf algebra $U_q(su(2))$ reduces to the envelopping algebra of $Lie(SU(2))$. Upon the standard identification

$$-ie = -R_1 + iR_2, \quad ie^* = R_1 + iR_2, \quad k = q^{iR_3}, \quad k^{-1} = q^{-iR_3},$$

we indeed obtain in the limit the standard definition of the $U(su(2))$ (viewed as the Hopf algebra) from the defining relations (12abc) of $U_q(su(2))$. In particular, the commutations relations (12a) gives in the limit $[R_j, R_k] = \epsilon_{jkl} R_l$. (Note that $R_j^* = -R_j$.) In the limit $q \rightarrow 1$, the action (13a) of $U_q(su(2))$ thus gives

$$(R_3 \triangleright B)_{q \rightarrow 1} = \lim_{q \rightarrow 1} \frac{k-1}{i \ln q} \triangleright B = iB, \quad (R_3 \triangleright B^*)_{q \rightarrow 1} = -iB^*, \quad (R_3 \triangleright A)_{q \rightarrow 1} = 0.$$

$$((R_1 + iR_2) \triangleright B)_{q \rightarrow 1} = \lim_{q \rightarrow 1} (ie^* \triangleright B) = i(2A - 1), \quad ((R_1 + iR_2) \triangleright B^*)_{q \rightarrow 1} = 0,$$

$$((-R_1 + iR_2) \triangleright B)_{q \rightarrow 1} = 0, \quad ((-R_1 + iR_2) \triangleright B^*)_{q \rightarrow 1} = \lim_{q \rightarrow 1} (-ie \triangleright B^*) = i(2A - 1),$$

$$((R_1 + iR_2) \triangleright A)_{q \rightarrow 1} = -iB^*, \quad ((-R_1 + iR_2) \triangleright A)_{q \rightarrow 1} = -iB. \quad (15a)$$

Comparing (15a) with (14a), we immediately observe that the $q \rightarrow 1$ limit of the $U_q(su(2))$ action on the Podleś sphere indeed coincide with the $Lie(SU(2))$ action induced by the projective formula.

Now we turn our attention to the q -deformation of the action of $Lie(AN)$. We define the following elements of $\text{Fun}_q(SU(2))$:

$${}^qT_0 = \frac{a - a^*}{2(\ln q)}, \quad i^qT_1 + {}^qT_2 = \frac{b}{(\ln q)}, \quad i^qT_1 - {}^qT_2 = \frac{b^*}{(\ln q)} \quad (16a)$$

and calculate

$$\lim_{q \rightarrow 1} ({}^qT_0 \triangleright B) = B(2A - 1), \quad \lim_{q \rightarrow 1} ({}^qT_0 \triangleright B^*) = B^*(2A - 1), \quad \lim_{q \rightarrow 1} ({}^qT_0 \triangleright A) = 2A(A - 1).$$

$$\lim_{q \rightarrow 1} (i^qT_1 + {}^qT_2) \triangleright B = 2B^2, \quad \lim_{q \rightarrow 1} (i^qT_1 + {}^qT_2) \triangleright B^* = -2A^2,$$

$$\lim_{q \rightarrow 1} ((i^qT_1 - {}^qT_2) \triangleright B) = 2A^2, \quad \lim_{q \rightarrow 1} (i^qT_1 - {}^qT_2) \triangleright B^* = -2B^{*2},$$

$$\lim_{q \rightarrow 1} (i^qT_1 + {}^qT_2) \triangleright A = 2AB, \quad \lim_{q \rightarrow 1} (i^qT_1 - {}^qT_2) \triangleright A = -2AB^*. \quad (15b)$$

Comparing (15b) with (14b), we immediately observe that the $q \rightarrow 1$ limit of the $\text{Fun}_q(SU(2))$ action on the Podleś sphere indeed gives the $\text{Lie}(AN)$ action induced by the projective formula (1).

The reader may find somewhat mysterious why the $q \rightarrow 1$ limit of $\text{Fun}_q(SU(2))$ contains $\text{Lie}(AN)$ generators. The explanation of this fact resides in the famous Drinfeld duality principle which states that there is a natural identification of Hopf algebras $\text{Fun}_q(G)$ and $U_q(\mathcal{G}^*)$. Here G is a Poisson-Lie group and \mathcal{G}^* is the Lie algebra of its dual Poisson-Lie group G^* . Let us indicate (a rigorous proof would require to give meaning to non-polynomial functions appearing in (16b)) why the Drinfeld duality takes place in the case $G = SU(2)$ and $G^* = AN$. The Lie algebra $\text{Lie}(AN)$ is generated by three generators T_j , $j = 0, 1, 2$, $T_j^* = -T_j$ obeying the following commutation relations:

$$[T_0, T_1] = -T_1, \quad [T_0, T_2] = -T_2, \quad [T_1, T_2] = 0. \quad (17)$$

We set

$$\begin{aligned} a &= q^{T_0} \sqrt{1 + q^2(\ln q)^2(T_1^2 + T_2^2)}, \quad a^* = \sqrt{1 + q^2(\ln q)^2(T_1^2 + T_2^2)} q^{-T_0}, \\ b &= (\ln q)(iT_1 + T_2), \quad b^* = (\ln q)(iT_1 - T_2). \end{aligned} \quad (16b)$$

Then it is not difficult to check two things: 1) the formulae (16b) and (17) imply the defining commutation relations (11a) of the Hopf algebra $\text{Fun}_q(SU(2))$; 2) it holds $\lim_{q \rightarrow 1} (q T_j) = T_j$.

Remark 1: Note that this explicit relation (16b) between $U_q(\text{Lie}(AN))$ and $\text{Fun}_q(SU(2))$ *degenerates* when $q \rightarrow 1$. This fact was important for establishing the limit $q \rightarrow 1$ of the Hopf adjoint action of $\text{Fun}_q(SU(2))$ on $\text{Fun}_q(S^2) \subset \text{Fun}_q(SU(2))$. Indeed, it appears superficially that in the $q \rightarrow 1$ limit, the algebra $\text{Fun}_q(SU(2))$ becomes commutative and the adjoint action trivial. This observation is too naive, however, and the explanation of the paradox resides in the degeneration of the relation (16b) between the sets of generators T_j and a, a^*, b, b^* in the limit $q \rightarrow 1$.

Remark 2: We have established the correct $q \rightarrow 1$ limit of the quantum double action (7ab) by performing the detailed calculations with the generators, relations etc. However, it is also possible to establish it on the conceptual level. First of all, the experts in Poisson-Lie groups and Hopf algebras know

that the $q \rightarrow 1$ of the adjoint action (7b) of $U_q(\mathcal{G}^*)$ on itself is indeed the dressing transformation of the Poisson-Lie group G by its dual Poisson-Lie group G^* (here $G = SU(2)$ and $G^* = AN$). The reader can find the detailed proof of this fact in the paper of [Korogodsky] . The conceptual proof of the correct $q \rightarrow 1$ limit of the formula (7a) is even simpler. Indeed:

The standard left action of the envelopping algebra $U(su(2))$ on $\text{Fun}(SU(2))$ is given by left derivations, i.e. if X is an element of $su(2)$ and $h(g)$ is in $\text{Fun}(SU(2))$ then we have

$$(X \triangleright h)(g) = \frac{d}{dt} h(e^{-tX} g)|_{t=0}. \quad (18)$$

Recall the coproduct and the counit of the Hopf algebra structure of the non-deformed $\text{Fun}(SU(2))$:

$$(\Delta h)(g_1, g_2) = h(g_1 g_2), \quad \varepsilon(h) = h(e),$$

where e is the group unit. Recall also that $S(X) = -X$ for $X \in su(2) \subset U(su(2))$. Finally note the standard formula for the pairing $\langle \cdot, \cdot \rangle$ between $X \in su(2)$ and $h \in \text{Fun}(SU(2))$:

$$-\langle X, h \rangle = \varepsilon(X \triangleright h).$$

Putting all these pieces of information together, we see that (18) can be written as

$$(X \triangleright h)(g) = \frac{d}{dt} h(e^{-tX} g)|_{t=0} = \langle S^{-1}(X), h' \rangle h''(g).$$

This is indeed the formula (7a) for $H = U(su(2))$ and $\tilde{H} = \text{Fun}(SU(2))$. In this way, we have verified that the action (7a) of $U_q(su(2))$ on $\text{Fun}_q(SU(2))$ (and, consequently on $\text{Fun}_q(S^2) \subset \text{Fun}_q(SU(2))$) has the correct $q \rightarrow 1$ limit, because it is well-known that the left action of $SU(2)$ on $S^2 = SU(2)/U(1)$ is induced by the projective formula (1) for $\bar{\alpha} = \delta, \bar{\gamma} = -\beta$.

4 Conclusions and outlook

We have constructed the q -Lorentz group extension of the natural action of $U_q(su(2))$ on the Podleś sphere and shown that it can be naturally interpreted as the q -deformation of the projective formula $z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$ describing

the global conformal transformation of the Riemann sphere. Our results are rather mathematical in nature but we believe that they can be used mainly in mathematical physics e.g. in further studies of braided field theories (cf. [Oeckl]) and also in studies of q -differential operators (cf. the q -Dirac operator by [Dąbrowski & Sitarz]) on the Podleś sphere. Indeed, our studies suggest to investigate the symmetry properties of those objects not only from the point of view of the action of the $U_q(su(2))$ quantum group but also from the point of view of the action of its quantum double.

Acknowledgement: I thank to R. Oeckl for discussions and to A. Sitarz for providing me the draft of his paper [Dąbrowski et al] prior to publication.

Note added: After having posted to the arch-ive the second version of this paper, S. Woronowicz has attracted my attention to his joint paper with W. Pusz in which they induced the representations of the q -Lorentz group from the characters of its parabolic subgroup. Among the representations constructed in this way there is also one that corresponds to the action of the q -Lorentz group on the Podleś sphere.

References:

- [Drinfeld] V. G. Drinfeld, *Quantum groups* in Proc. ICM, MSRI, Berkeley, 1986;
- [Dąbrowski & Sitarz] L. Dąbrowski and A. Sitarz, *Dirac operator on the standard Podleś quantum sphere*, Noncommutative geometry and quantum groups (Warsaw, 2001), 49-58, Banach Center Publ., 61, Polish Acad. Sci., Warsaw, 2003, math.QA/0209048,
- [Dąbrowski et al] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. Varilly, *The Dirac operator on $SU_q(2)$* , to appear
- [Kassel] C. Kassel, *Quantum groups*, Springer-Verlag, New York, 1995;
- [Klimčík] C. Klimčík, *Quasitriangular WZW model*, Sec. 4.1, Rev. Math. Phys. **16**, (2004) 679 - 808, hep-th/0103118
- [Korogodsky] L. Korogodsky, *Complementary series representations and quantum orbit method*, q-alg/9708026;
- [Levendorskii & Soibelman] S. Levendorskii and Y. Soibelman, *Algebras of functions on compact quantum groups, Schubert cells and quantum tori*, Commun. Math. Phys. **139** 141 - 169 (2001);
- [Majid] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995;
- [Oeckl] R. Oeckl, *Braided quantum field theory*, Commun. Math. Phys. **217** 451 - 473 (2001);
- [Podleś] P. Podleś, *Quantum spheres*, Lett. Math. Phys. **14** 521 - 531 (1987);
- [Podleś & Woronowicz] P. Podleś and S. Woronowicz, *Quantum deformation of Lorentz group*, Commun. Math. Phys. **130** 381 - 431 (1990);
- [Pusz & Woronowicz] W. Pusz and S. Woronowicz, *Representations of Quantum Lorentz Group on Gelfand spaces*, Rev. Math. Phys. **12** 1551-1625 (2000)
- [Semenov-Tian-Shansky] M. A. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions*, Publ. RIMS, Kyoto University **21** 1237 - 1260 (1985).